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Logicoalgebraic Approach to Lagrangian Systems

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A logicoalgebraic approach to the geometry of Lagrangian systems is pursued by starting axiomatically with a classical mechanical system whose logic is a separable and atomic Boolean σ algebra.

1. INTRODUCTION

The logicoalgebraic approach to classical mechanics conceives the logic of all experimentally verifiable propositions concerning a mechanical system Σ as a Boolean σ algebra $\mathcal{L}$; states and observables are then described as probability measures on $\mathcal{L}$ and σ homomorphisms from the Borel structure of real line R onto $\mathcal{L}$, respectively (see Varadarajan, 1968 and 1970, for relevant algebraic definitions). We shall be precisely concerned with a separable and atomic Boolean σ algebra $\mathcal{L}$. Separability indeed corresponds to finiteness of degrees of freedom, i.e., to existence of a finite complete system of observables (Kronfli, 1970). Atomicity then corresponds to determinism, i.e., to existence of a subset $\mathcal{P}$ of states in which strict values of all observables can be checked (Barone and Galdi, 1979); $\mathcal{P}$ turns out to be the set of deterministic or pure states—probability measures with 0 and 1 values only—which may be as well characterized as measures concentrated at atoms of $\mathcal{L}$.

On the other hand, the Lagrangian-geometrical approach to classical mechanics describes a deterministic system with finite degrees of freedom Σ as a Lagrangian system (Q, L), where Q is a differentiable configuration

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space (defined by constraint equations) and L is a Lagrangian function on tangent bundle TQ (difference between kinetic and potential energies), which defines an isomorphic Legendre transformation from TQ to cotangent bundle T^*Q. States and observables are then described as points of, and real-valued functions on, phase space TQ, or T^*Q (see Abraham and Marsden, 1978, for relevant geometrical definitions).

Now the question arises whether, starting axiomatically with a separable and atomic Boolean σ algebra $\mathfrak{L}$, one can achieve the above theory of Lagrangian systems.

(i) Let us first remark that transition from the algebraic to geometrical point of view necessarily requires a set-theoretical representation of abstract algebra $\mathfrak{L}$.

As is well known, the whole representation theory of Boolean σ algebras is essentially held by the fundamental Loomis theorem (Loomis, 1947), which, for a separable Boolean σ algebra $\mathfrak{L}$, comes out to claim the existence of a (nonunique) standard Borel space with a Borel structure σ epimorphic to $\mathfrak{L}$ (Varadarajan, 1968).

$\mathfrak{L}$ being atomic too, Loomis theorem may be further specialized and the result is a representation theorem (Section 2) which claims the existence of a (nonunique) separable Borel space with a Borel structure σ isomorphic to $\mathfrak{L}$.

Any such space gives pure states and observables of $\mathfrak{L}$ the set-theoretical description we just expect of a phase space, and will be then called a *Borel phase space* of $\mathfrak{L}$.

(ii) Let us then remark that this array of Borel phase spaces cannot be reduced to a single one whose geometry does correspond to kinematics of Σ, as long as any preferred structure of observables is not explicitly associated with $\mathfrak{L}$. Kinematics indeed—i.e., physical-geometrical description of states of Σ—is not to be thought of as an intrinsic feature of Σ, uniquely exhibited by its logic $\mathfrak{L}$, but just as an additional structure of observables to be associated with $\mathfrak{L}$, whose choice is based upon observer's physical-geometrical criterions.

In this connection (Section 3) any equivalence class H of complete systems of fundamental observables (position and velocity) will be called a *kinematics* on $\mathfrak{L}$.

If $u \in H$, then a Borel phase space P which characterizes states of $\mathfrak{L}$ in terms of strict values of u and any other observable of $\mathfrak{L}$ as a function of these values, is uniquely determined.

(iii) The above result allows us (Section 4) to introduce a classical holonomy condition on H, by means of suitable constraint equations on strict values of u; a differentiable configuration space Q is so associated with $\mathfrak{L}$ and P identified with velocity phase space TQ (in this case P, as a Borel space, is standard and then $\mathfrak{L}$ will be said to be standard too).

A Lagrangian observable Λ, satisfying a strong hyperregularity condition, is then able to identify P with momentum phase space T^*Q.

Both holonomy and hyperregularity conditions—together with consequent bundle structures on P—are shown to be invariant under kinematically equivalent transformations of fundamental observables u.

Then a standard logic $\mathfrak{L}$ endowed with a holonomic kinematics H and a hyperregular Lagrangian Λ, will be the final logicoalgebraic definition of *Lagrangian system*.

2. BOREL PHASE SPACE

Let $\mathfrak{L}$ be a separable and atomic Boolean σ algebra (whose partial order, zero and meet are $<$, $\mathbf{0}$, and $\wedge$, respectively).

Owing to separability, the Loomis representation theorem may be stated as follows. There exists a σ epimorphism

$$u: \mathfrak{B}(X) \to \mathfrak{L} \tag{1}$$

from the Borel structure $\mathfrak{B}(X)$ of a standard Borel space X onto $\mathfrak{L}$; X stands for R [Varadarajan, 1968, p. 17, Theorem 1.6(i)] or any other standard Borel space with the power of the continuum, all of them being Borel isomorphic (Varadarajan, 1970, p. 3).

Owing to atomicity, the Loomis theorem will be further specialized as follows.

Let

$$P = \{x \in X / u(\{x\}) \neq \mathbf{0}\} \tag{2}$$

be the subset of X that u bijectively maps onto the set of atoms of $\mathfrak{L}$ (Kronfli, 1970, p. 396, Theorem 2).

Theorem 1. There exists a σ isomorphism

$$\varphi: \mathfrak{B}(P) \to \mathfrak{L} \tag{3}$$

from the Borel structure $\mathfrak{B}(P) = (E \cap P)_{E \in \mathfrak{B}(X)}$ of P onto $\mathfrak{L}$.

Proof. Atomicity of $\mathfrak{L}$ entails the following lemma.

For any two $E_1, E_2 \in \mathfrak{B}(X)$, $E_1 \cap P = E_2 \cap P$ iff $u(E_1) = u(E_2)$.

In fact, remark that (if E stands for $E_1 - E_2$, or $E_2 - E_1$)$x \in E \cap P$ iff $\mathbf{0} \neq u(\{x\}) < u(E)$, and then $E \cap P \neq \varnothing$ iff $u(E) \neq \mathbf{0}$.

The above lemma entails the existence of a natural σ isomorphism

$$u_1 : \mathfrak{B}(P) \to \mathfrak{B}(X)_{/\mathrm{Ker}(u)}$$

from $\mathfrak{B}(P)$ onto the quotient of $\mathfrak{B}(X)$ modulo the kernel of u.

Then, if σ isomorphism

$$u_2 : \mathfrak{B}(X)_{/\mathrm{Ker}(u)} \to \mathfrak{L}$$

is the quotient of u modulo Ker(u), σ isomorphism (3) is given by

$$\varphi = u_2 \circ u_1 \qquad \blacksquare \qquad (3')$$

P is a separable Borel space [it is standard iff $P \in \mathfrak{B}(X)$; see Varadarajan, 1970, pp. 2 and 3].

σ Isomorphism φ bijectively takes any point x of P to pure state p of $\mathfrak{L}$ given by the probability measure concentrated at atom $\varphi(\{x\})$, and any real-valued Borel function f on P to observable v of $\mathfrak{L}$ given by $v = \varphi \circ f^{-1}$; the result is $p(v(\{f(x)\})) = 1$, i.e., $f(x)$ is the (unique) strict value of v in p.

So pure states and observables of $\mathfrak{L}$ are characterized as points of, and functions on P, just as we expect of a phase space.

We are then led to state the following:

Definition 1. A separable Borel space whose Borel structure is σ isomorphic to $\mathfrak{L}$, is called a *Borel phase space* of $\mathfrak{L}$.

Theorem 1 shows the existence of many different Borel phase spaces of $\mathfrak{L}$.

3. KINEMATICS

Let u be a σ epimorphism (1), with

$$X = R^N$$

It characterizes a complete system of observables, given by

$$u_i = u \circ p_i^{-1} \qquad (i = 1, \ldots, N) \qquad (1')$$

where p_i is the ith coordinate projection of R^N onto R [Varadarajan, 1968, pp. 17 and 18, Theorem 1.6(ii)].

Let P be the Borel phase space associated with u by equation (2).

It characterizes pure states of $\mathfrak{L}$ in terms of the strict values of observables (1′) and any other observable of $\mathfrak{L}$ as a function of these values, according to the following:

Theorem 2. The following equality holds true:

$$P=\{(x_i)\in R^N/\exists p\in\mathcal{P}: p(u_i(\{x_i\}))=1\ (i=1,\ldots,N)\}$$

Proof. For any $x=(x_i)\in R^N$, $u(\{x\})=\wedge_{i=1}^{N}u_i(\{x_i\})$.

Therefore, if $u(\{x\})\neq\mathbf{0}$, then probability measure $p(u(\{x\}))=1$ concentrated at atom $u(\{x\})$ satisfies condition $p(u_i(\{x_i\}))=1$ $(i=1,\ldots,N)$.

Conversely, if $p(u_i(\{x_i\}))=1$ $(i=1,\ldots,N)$, then $p(u(\{x\}))=1$ and $u(\{x\})\neq\mathbf{0}$. ■

In particular, if

$$R^N=TR^n$$

observables $(u_i), i\in\{1,\ldots,n\}$, whose strict values are coordinates in the base space of TR^n, will be called *position observables*; observables (u_j), $j\in\{n+1,\ldots,2n=N\}$, whose strict values are coordinates in the standard fibre of TR^n, will be called *velocity observables*.

According to Theorem 2, P will have to be regarded as the set of all possible values of position and velocity coordinates.

On the other hand, any diffeomorphism k of R^n defines—through tangent automorphism Tk—a transformation of position and velocity coordinates, which corresponds to the transformation of position and velocity observables $u'=u\circ Tk^{-1}$.

This defines a kinematic equivalence relation K among complete systems of observables (of even order).

We are then led to state the following:

Definition 2. Any K equivalence class H is called a *kinematics* on $\mathfrak{L}$.

4. LAGRANGIAN SYSTEM

Let H be a kinematics on $\mathfrak{L}$.

Then let P be the Borel phase space associated with a system $u\in H$.

Assume that

(i) $$P=\mathrm{Ker}(Tf)$$

where Tf is the tangent morphism of a differentiable map $f: R^n\to R^s (s<n)$ that, if nonvanishing, has a derivative of maximal rank on

$$Q=\mathrm{Ker}(f)$$

Holonomy condition (i) corresponds to the existence of constraint equations involving only position coordinates, which define a differentiable configuration space Q; equations inferred from the previous ones by derivation then identify P with velocity phase space TQ (see, e.g., Abraham and Marsden, 1978, p. 49).

Theorem 3. Holonomy condition (i) is K invariant.

Proof. It is enough to remark that, if $u' = u \circ Tk^{-1} \in H$, then Tk maps P onto P' (the Borel phase space associated with u') and $\mathrm{Ker}(Tf)$ onto $\mathrm{Ker}(Tf')$, with $f' = f \circ k^{-1}$. ■

We may then state the following:

Definition 3. H is called a *holonomic kinematics* on $\mathfrak{L}$ if condition (i) holds true.

Remark that, if H is a holonomic kinematics, Borel phase space P associated with $u \in H$ is standard (in fact, owing to condition (i), $P \in \mathfrak{B}(TR^n)$).

Then only *standard logics*, i.e., Boolean σ algebras which admit of standard Borel phase spaces, will be now taken into account.

Let H be a holonomic kinematics on a standard logic $\mathfrak{L}$, and Λ an observable associated with $\mathfrak{L}$ called *Lagrangian*.

Then, if $u \in H$ and φ is the σ isomorphism inferred from u by equation (3′), let

$$L: P \to R$$

be the Borel function characterized by $\Lambda = \varphi \circ L^{-1}$, and, when L is differentiable,

$$FL: P \to T^*Q$$

its fibre derivative (which is a bundle morphism from $P = TQ$ to T^*Q; see Abraham and Marsden, 1978, p. 209).

Assume the following:

(ii) L is a differentiable function, and FL is a vector bundle isomorphism.

Strong hyperregularity condition (ii) corresponds to a Legendre transformation FL which identifies P with momentum phase space T^*Q (Abraham and Marsden, 1978, p. 223).

Theorem 4. Hyperregularity condition (ii) is K invariant.

Proof. Let $u' = u \circ Tk^{-1} \in H$.

Then σ isomorphism φ' inferred from u' by equation (3′) is related to φ by

$$\varphi' = \varphi \circ T\xi^{-1}$$

where $\xi = k_{|Q}$.

As a consequence, Borel function $L' : P' \to R$ characterized by $\Lambda = \varphi' \circ L'^{-1}$ is related to L by

$$L' = L \circ T\xi^{-1}$$

This entails differentiability of L' and commutativity of diagram

$$\begin{array}{ccc} P & \xrightarrow{FL} & T^*Q \\ T\xi \downarrow & & \uparrow \xi^* \\ P' & \xrightarrow[FL']{} & T^*Q' \end{array}$$

[where $Q' = \mathrm{Ker}(f')$ and ξ^* is the transpose of ξ]. ∎

We may then state the following:

Definition 4. Λ is called a *hyperregular Lagrangian* on $(\mathcal{L}, H)$ if condition (ii) holds true.

Therefore the following logicoalgebraic definition of Lagrangian system is achieved.

Definition 5. A *Lagrangian system*

$$\Sigma = (\mathcal{L}, H, \Lambda)$$

is given by a standard logic $\mathcal{L}$, a holonomic kinematics H on $\mathcal{L}$, and a hyperregular Lagrangian Λ on $(\mathcal{L}, H)$.

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